Algebraic Curves and Abelian Varieties

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Chapter 1

Classical Varieties

Varieties (or *algebraic varieties*) are spaces cut out by polynomials: they are the vanishing loci of a family of multivariate polynomials. We demonstrate the usefulness of this idea with the following simple number theory question, before we introduce the notion formally.

Question 1.0.1. What are the integers solutions $(a, b, c) \in \mathbb{Z}^3$ to the equation $a^2 + b^2 = c^2$? It suffice to find all coprime solutions (a, b, c) since other solutions are multiples of them.

The most natural solution of this question comes from viewing the equation geometrically. Indeed, if we assume $c \neq 0$, then dividing c^2 on both sides of $a^2 + b^2 = c^2$ we get $\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$, so the points $(x, y) = \left(\frac{a}{c}, \frac{b}{c}\right)$ lie on the unit circle. So the problem becomes, how does one find all rational points (i.e. points with rational coordinates) on the unit circle?

Appendix A

Appendix