Arithmetic Monodromy and Middle Convolution of ℓ-adic Perverse Sheaves

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Yunhai Xiang

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1 Introduction

1.1 Notations

1.1.1. Motivated by the growing interest in the formalization of mathematics within the mathematical community, we adopt a style of notation inspired by type theory throughout this report, without committing to type-theoretic foundations. The symbols below denote types and/or categories, depending on your philosophical inclination. The notation $X : \mathsf{T}$ means that X is a term of type T or an object of the category T.

Set	Sets
$\mathsf{Set}^{\mathrm{fin}}$	Finite sets
Top	Topological spaces
$\mathsf{Open}(T)$	Open subsets of $T : \mathsf{Top}$
Grp	Groups
$\mathsf{Grp}^{\mathrm{fin}}$	Finite groups
$\mathsf{Grp}^{\mathrm{top}}$	Topological groups
$\mathsf{Grp}^{\mathrm{pro}}$	Profinite groups
Ring	Rings (associative, unital, not necessarily commutative)
$\mathsf{Ring}^{\mathrm{top}}$	Topological rings
Field	Fields
Alg_k	Commutative unital algebras over $k : \mathsf{Field}$
Mod_R	Left modules over $R : \mathsf{Ring}$
$\mathsf{Mod}_R^{\mathrm{top}}$	Topological left modules over $R : \mathsf{Ring}^{\mathrm{top}}$
Vect_k	Vector spaces over $k : \mathsf{Field}$
$\mathsf{Vect}_k^{\mathrm{fin}}$	Finite-dimensional vector spaces over $k : \mathsf{Field}$
$\mathsf{Rep}_\Lambda(G)$	Continuous Λ-linear representations of $G : \mathsf{Grp}^{\mathrm{top}}$, where $\Lambda : \mathsf{Ring}^{\mathrm{top}}$
$\mathsf{Rep}_\Lambda^{\mathrm{fin}}(G)$	Continuous Λ-linear representations of $G : \mathsf{Grp}^{\mathrm{top}}$, where $\Lambda : \mathsf{Ring}^{\mathrm{top}}$, whose underlying topological Λ-modules are finitely generated
Sch	Schemes
Sch_S	Schemes over a base scheme $S : \mathsf{Sch}$
Var_k	Varieties over $k : \mathsf{Field}$, i.e. integral separated schemes of finite type over k
Curv_k	Curves over $k : \mathsf{Field}$, i.e. one-dimensional varieties over k
$\mathsf{Curv}_k^{\mathrm{sm}}$	Smooth curves over $k : \mathsf{Field}$
$\mathsf{Man}_{\mathbb{C}}$	Complex manifolds
Riem	Riemann surfaces, i.e. connected one-dimensional complex manifolds
Site	Sites, i.e. categories equipped with Grothendieck topologies
$\mathsf{Sh}(X, \mathsf{A})$	Sheaves on $X : \mathsf{Site}$ with values in a category A admitting small limits
$\mathsf{D}^{\mathrm{b}}(\mathsf{A})$	Bounded derived objects of an abelian category A
$X \to Y$	Morphisms from $X : \mathsf{T}$ to $Y : \mathsf{T}$

1.2 Acknowledgements

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1.2.2 Typesetting. The typesetting style of this report is inspired by [Kat88].

1.2.3 Declaration of AI Use. The use of generative AI in the preparation of this report was limited to the following purposes: proofreading; polishing wording, phrasing, grammar, and formatting; helping with LaTeX typesettings; searching for and citing references; suggesting writing advices; answering mathematical questions from the author; translating reading materials in French into English; and retypesetting old references with archaic typesettings for ease of reading.

1.3 Motivations

1.3.1. *Monodromy groups* are interesting mathematical objects that arise naturally in both geometric and arithmetic contexts. Roughly speaking, they describe how locally defined data change when transported along a loop. Controlling and understanding monodromy groups has been a central theme in both of these worlds, with significant applications.

1.3.2. The goal of this report is to provide an exposition on monodromy groups as well as ways to study them using methods from perverse sheaves and their middle convolution introduced by Katz [Kat96], especially in the ℓ-adic arithmetic context.

1.3.3. Consider the equation

$$z(u)^2 = u \tag{1.3.3.1}$$

for the unknown holomorphic function $z : V \to \mathbb{C}^\times$, where $V : \mathsf{Open}(\mathbb{C}^\times)$ and $u : V$ is the independent variable. Fix a base point $u_0 : \mathbb{C}^\times$ and choose $z_0 : \mathbb{C}^\times$ with $z_0^2 = u_0$. On a sufficiently small disc V containing u_0, the equation has two holomorphic solution branches $z_+(u) = \sqrt{u}$ and $z_-(u) = -\sqrt{u}$ with $z_+(u_0) = z_0$ and $z_-(u_0) = -z_0$. Move u once around 0 along the loop

$$\gamma : [0,1] \to \mathbb{C}^\times, \qquad t \mapsto u_0 e^{2\pi i t}.$$

Along this loop, the values of the two solution branches vary continuously as $\widetilde{\gamma}_+(t) = z_0 e^{\pi i t}$ and $\widetilde{\gamma}_-(t) = -z_0 e^{\pi i t}$, until they eventually swap places at $t = 1$. In other words, there is a permutation representation of the fundamental group[1] $\pi_1^{\mathrm{top}}(\mathbb{C}^\times, u_0) \cong \mathbb{Z} : \mathsf{Grp}^{\mathrm{top}}$

$$\rho : \pi_1^{\mathrm{top}}(\mathbb{C}^\times, u_0) \to \mathrm{Aut}_{\mathsf{Set}}(\{[z_+]_{u_0}, [z_-]_{u_0}\}), \qquad n \mapsto \left([z_+]_{u_0}\ [z_-]_{u_0}\right)^n.$$

We call this representation the *permutation monodromy representation*, and we call its image, isomorphic to S_2, the *permutation monodromy group*, of the equation (1.3.3.1) at u_0.

1.3.4. We reformulate the above with sheaves. Consider the covering

$$f : \mathbb{C}^\times \to \mathbb{C}^\times, \qquad \zeta \mapsto \zeta^2.$$

Let $\mathcal{F} : \mathsf{Sh}(\mathbb{C}^\times, \mathsf{Set})$ be its sheaf of holomorphic sections. Equivalently, this is the sheaf of solutions to the equation (1.3.3.1),

$$\begin{aligned}\mathcal{F}(V) &= \{z : V \to \mathbb{C}^\times \mid z \text{ holomorphic and } (f \circ z)|^V = \mathrm{id}_V\} \\ &= \{z : V \to \mathbb{C}^\times \mid z \text{ holomorphic and } z(u)^2 = u \text{ for every } u : V\}.\end{aligned}$$

[1] with the discrete topology

As in 1.3.3, on a sufficiently small disc V containing u_0, the sections are precisely the two holomorphic solution branches z_+ and z_-, identifying

$$\mathcal{F}|_V \cong \underline{\{+,-\}}_V.$$

Hence $\mathcal{F}$ is locally constant, with stalk at u_0

$$\mathcal{F}_{u_0} = \{[z_+]_{u_0}, [z_-]_{u_0}\} \cong f^{-1}(u_0)$$

identified by evaluation. However, the sheaf is not constant: $\mathcal{F}(\mathbb{C}^\times) = \emptyset$. A global section would determine a germ at u_0 fixed by all monodromy at u_0, whereas transport once around 0 exchanges the two square-root germs, as demonstrated in 1.3.3. Locally constant (but not necessarily constant) sheaves of sets, such as this, are known as *local systems of sets*.

1.3.5. For a path $\gamma : [0,1] \to \mathbb{C}^\times$ from $u_0 = \gamma(0)$ to $u_1 = \gamma(1)$ and a point $\zeta : f^{-1}(u_0)$ in the fibre, let $\widetilde{\gamma}_\zeta$ be the unique lift through f satisfying $\widetilde{\gamma}_\zeta(0) = \zeta$. We have the *transport map*

$$T_\gamma : \mathcal{F}_{u_0} \xrightarrow{\sim} \mathcal{F}_{u_1}, \qquad \zeta \longmapsto \widetilde{\gamma}_\zeta(1).$$

with the evaluation identifications $\mathcal{F}_u \cong f^{-1}(u)$. Thus we obtain a *transport functor*[2]

$$T : \Pi_1(\mathbb{C}^\times) \longrightarrow \mathsf{Set}, \qquad u \longmapsto \mathcal{F}_u, \qquad [\gamma] \longmapsto T_\gamma.$$

Restricting transport to loops at u_0 gives the permutation monodromy representation of $\mathcal{F}$ at u_0, the same permutation monodromy representation as in 1.3.3

$$\rho_{\mathcal{F},u_0} : \pi_1^{\mathrm{top}}(\mathbb{C}^\times, u_0) \to \mathrm{Aut}_{\mathsf{Set}}(\mathcal{F}_{u_0}), \qquad [\gamma] \mapsto T_\gamma.$$

1.3.6. In § 2, we will see how to upgrade permutation monodromy representations into linear monodromy representations, which we will just call monodromy representations. Moreover, one can use this machinery to talk about monodromy in the ℓ-adic arithmetic context, by replacing π_1^{top} with the étale fundamental group $\pi_1^{\text{ét}}$. When $U = \mathrm{Spec}(k)$ for some k : Field, with a choice of separable closure k^{sep}, then $\pi_1(U, u_0) = \mathrm{Gal}(k^{\mathrm{sep}}/k)$, where $u_0 = \mathrm{Spec}(k^{\mathrm{sep}})$, is the absolute Galois group of k. Hence, a monodromy representation at u_0 in this case degenerates into a Galois representation, a ubiquitous arithmetic object.

2 Local Systems

2.1 Complex Local Systems

2.1.1. We have looked at permutation monodromy, but permutation actions do not capture all the situations we wish to study, since a lot of times in nature we encounter actions on vector spaces instead of sets. For example, instead of an algebraic equation like the one in 1.3.3, we consider a differential equation, which is known as *Gauss's hypergeometric equation*

$$u(1-u)\frac{\mathrm{d}^2 z}{\mathrm{d}u^2} + \big(c - (a+b+1)u\big)\frac{\mathrm{d}z}{\mathrm{d}u} - abz = 0 \tag{2.1.1.1}$$

[2]Here $\Pi_1(X)$ denotes the fundamental groupoid of some X : Top, that is, the category whose objects are points of X and whose morphisms are homotopy classes of paths relative to their endpoints, with concatenation as composition.

where $a, b, c : \mathbb{C}$ are fixed parameters and $z : V \to \mathbb{C}$ is an unknown holomorphic function, with $V : \mathsf{Open}(U)$, $U = \widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$, and independent variable $u : V$. We then define the sheaf of solutions to this equation $\mathcal{F} : \mathsf{Sh}(U, \mathsf{Vect}_{\mathbb{C}})$

$$\mathcal{F}(V) = \{z : V \to \mathbb{C} \mid z \text{ holomorphic and satisfies (2.1.1.1) on } V\}.$$

This is a sheaf of vector spaces since (2.1.1.1) is a homogeneous linear differential equation. For each $u_0 : U$, the existence-and-uniqueness theorem [HTT08, Theorem 4.3.1] gives a sufficiently small open disc $V : \mathsf{Open}(U)$ containing u_0 and an isomorphism of sheaves

$$\mathcal{F}|_V \xrightarrow{\sim} \underline{\mathbb{C}}^2_V$$

Thus $\mathcal{F}$ is locally constant of rank 2. We call such a sheaf a *complex local system of rank* 2.

2.1.2 Definition. (Complex Local System) Let $U : \mathsf{Man}_{\mathbb{C}}$ be connected. A *complex local system* on U is a locally constant sheaf $\mathcal{F} : \mathsf{Sh}(U, \mathsf{Vect}_{\mathbb{C}})$ with finite-dimensional stalks. Explicitly, every $u : U$ has an open neighbourhood $V : \mathsf{Open}(U)$ and an isomorphism

$$\varphi_V : \mathcal{F}|_V \xrightarrow{\sim} \underline{\mathbb{C}}^n_V$$

called a *local trivialization* at u, for some $n : \mathbb{N}$ called the *rank* of $\mathcal{F}$. Denote by $\mathsf{LocSys}_{\mathbb{C}}(U)$ the (symmetric monoidal $\mathbb{C}$-linear) category of complex local systems on U.

2.1.3 Definition. (Transport) Let $\mathcal{F}$ be a complex local system of rank n on a connected $U : \mathsf{Man}_{\mathbb{C}}$. We define the *transport functor* using analytic continuation

$$T : \Pi_1(U) \to \mathsf{Vect}^{\mathrm{fin}}_{\mathbb{C}}, \qquad u \mapsto \mathcal{F}_u, \qquad [\gamma] \mapsto T_\gamma.$$

where the *transport map* $T_\gamma : \mathcal{F}_{\gamma(0)} \xrightarrow{\sim} \mathcal{F}_{\gamma(1)}$ for a path $\gamma : [0, 1] \to U$ is defined as follows. By compactness of closed intervals, we can choose a subdivision $0 = t_0 < \cdots < t_N = 1$ and open sets $V_i : \mathsf{Open}(U)$ containing $\gamma([t_{i-1}, t_i])$, with local trivializations

$$\varphi_i : \mathcal{F}|_{V_i} \xrightarrow{\sim} \underline{\mathbb{C}}^n_{V_i}.$$

Put $v_i = \gamma(t_i)$ for each $i = 0, \ldots, N$, then the trivializations induce stalk isomorphisms $\varphi_{i,v_i} : \mathcal{F}_{v_i} \xrightarrow{\sim} \mathbb{C}^n$. The transport map along γ is then defined as the composite

$$T_\gamma = \left(\varphi^{-1}_{N,v_N} \circ \varphi_{N,v_{N-1}}\right) \circ \cdots \circ \left(\varphi^{-1}_{1,v_1} \circ \varphi_{1,v_0}\right) : \mathcal{F}_{\gamma(0)} \xrightarrow{\sim} \mathcal{F}_{\gamma(1)}.$$

which is independent of the subdivision and trivializations and depends only on the homotopy class of γ relative to its endpoints, by monodromy theorem [Sch11, Lemma 7.2.7].

2.1.4 Definition. (Monodromy) Let $\mathcal{F}$ be a complex local system of rank n on a connected $U : \mathsf{Man}_{\mathbb{C}}$, and fix a base point $u_0 : U$. Restricting its transport functor to loops at u_0 gives the *monodromy representation* of $\mathcal{F}$ at u_0

$$\rho_{\mathcal{F},u_0} : \pi^{\mathrm{top}}_1(U, u_0) \to \mathrm{Aut}_{\mathsf{Vect}_{\mathbb{C}}}(\mathcal{F}_{u_0}), \qquad [\gamma] \mapsto T_\gamma,$$

whose image is the *monodromy group* of $\mathcal{F}$ at u_0.

2.1.5. Like in 1.3.4, $\mathcal{F}$ is also a sheaf of sections of a cover. The *espace étalé* of U is

$$\text{Ét}(U) := \coprod_{u \in U} \mathcal{F}_u : \mathsf{Man}_{\mathbb{C}}$$

with topology generated by sets $\widetilde{s}(V) = \{(u, s_u) : u : V\}$ for $V : \mathsf{Open}(U)$ and $s : \mathcal{F}(V)$. Each basic open $\widetilde{s}(V)$ is identified with V by $(u, s_u) \mapsto u$, so the complex charts of U pull back to a compatible complex atlas on $\text{Ét}(U)$.[3] Its associated covering map

$$f : \text{Ét}(U) \to U, \qquad (u, \xi) \mapsto u$$

is the analogue of the double cover in 1.3.4, and its sheaf of sections recovers $\mathcal{F}$. Analytic continuation on U is equivalent to path lifting to the espace étalé, so the transport can also be defined equivalently via path lifting, such as in 1.3.5.

2.1.6. In the other direction, if we are given a representation

$$\rho : \pi_1^{\text{top}}(U, u_0) \to \text{Aut}_{\mathsf{Vect}_{\mathbb{C}}}(W)$$

for some connected $U : \mathsf{Man}_{\mathbb{C}}$, base point $u_0 : U$ and $W : \mathsf{Vect}_{\mathbb{C}}^{\text{fin}}$ of dimension n, we can recover a local system $\mathcal{F}$ of rank n on U whose monodromy representation is ρ by the following way. Pick a universal cover $q : \widetilde{U} \to U$ and a basepoint lift $\widetilde{u_0} : q^{-1}(u_0)$. Define a right $\pi_1^{\text{top}}(U, u_0)$ action on $\widetilde{U} \times W_{\text{disc}}$, where W_{disc} has the discrete topology, by

$$(\widetilde{u}, w) \cdot g = (\widetilde{u} \cdot g, \rho(g)^{-1} w).$$

where $\widetilde{u} \cdot g$ is the action by the deck transformation determined by $\widetilde{u_0}$. Form the orbit space

$$E_\rho = (\widetilde{U} \times W_{\text{disc}}) / \pi_1^{\text{top}}(U, u_0)$$

We note that E_ρ inherits a complex structure induced by its associated covering

$$f_\rho : E_\rho \to U, \qquad [\widetilde{u}, w] \mapsto q(\widetilde{u})$$

for which it is a local biholomorphism i.e. étale cover. These recover the espace étalé of our desired local system and its associated cover. Therefore, we can take its sheaf of sections $\mathcal{F}_\rho$ to obtain our desired local system, where for each $V : \mathsf{Open}(U)$

$$\begin{aligned} \mathcal{F}_\rho(V) &= \{s : V \to E_\rho \mid s \text{ holomorphic}^4 \text{ and } (f_\rho \circ s)|^V = \text{id}_V\} \\ &\cong \left\{ a : q^{-1}(V) \to W_{\text{disc}} \;\middle|\; \begin{array}{l} a \text{ locally constant,} \\ a(\widetilde{u} \cdot g) = \rho(g)^{-1} a(\widetilde{u}) \end{array} \right\}. \end{aligned}$$

where the last identification is given by $s(q(\widetilde{u})) = [\widetilde{u}, a(\widetilde{u})]$ for every $\widetilde{u} : q^{-1}(V)$.

2.1.7. From 2.1.6, we see that for each $U : \mathsf{Man}_{\mathbb{C}}$ and $u_0 : U$, there are symmetric monoidal $\mathbb{C}$-linear equivalences of categories [Mac90, § 7.2]

$$\mathsf{LocSys}_{\mathbb{C}}(U) \simeq^{\otimes} \mathsf{Rep}_{\mathbb{C}}^{\text{fin}}\big(\pi_1^{\text{top}}(U, u_0)\big).$$

[3] Throughout this report, complex manifolds are not required to be second-countable. In particular, the *espace étalé* of a positive-rank complex local system has uncountable discrete fibres and hence is not second-countable.

[4] Continuity alone suffices: since f_ρ is a local biholomorphism, every continuous section is locally a holomorphic inverse of f_ρ, and hence is holomorphic.

2.1.8 Example. (Local Systems from Complex Families) A rich source of local systems is smooth and proper families. Let

$$f : E \to U$$

be a smooth proper map between complex manifolds. The fibres $E_{u_0} := f^{-1}(u_0)$ at $u_0 : U$ are compact complex manifolds. The higher direct image sheaves

$$\mathbf{R}^q f_* \underline{\mathbb{C}}$$

are complex local systems for each $q : \mathbb{N}$ by Ehresmann's fibration theorem [Voi02, Theorem 9.3]. Explicitly, at $u_0 : U$, there exists open neighborhood $V : \mathsf{Open}(U)$ of u_0 with

$$\mathbf{R}^q f_* \underline{\mathbb{C}}|_V \cong \underline{\mathrm{H}^q(E_{u_0}, \underline{\mathbb{C}})}_V$$

In particular on the level of stalks $\left(\mathbf{R}^q f_* \underline{\mathbb{C}}\right)_{u_0} \cong \mathrm{H}^q(E_{u_0}, \underline{\mathbb{C}})$.

2.2 ℓ-adic Local Systems: Lisse Sheaves

2.2.1. We now develop the ℓ-adic analogues of the constructions in § 2.1.

2.2.2. In number theory, ℓ-adic Galois representations are ubiquitous and encode a lot of intricate arithmetic information. Let $E : \mathsf{Curv}_k^{\mathrm{sm}}$ be an elliptic curve over a field $k : \mathsf{Field}$, recall that its Tate module at a prime $\ell : \mathbb{N}$ is the $\mathbb{Z}_\ell[\Gamma_k]$-module

$$\mathrm{T}_\ell(E) = \varprojlim E[\ell^n]$$

where $\Gamma_k = \mathrm{Gal}(k^{\mathrm{sep}}/k)$. In other words, it is an (continuous) ℓ-adic Galois representation

$$\rho_{E,\ell} : \Gamma_k \to \mathrm{Aut}_{\mathbb{Z}_\ell}(\mathrm{T}_\ell(E))$$

Theorems of Faltings and Zarhin–Mori show that the rationalized Tate module

$$\mathrm{V}_\ell(E) := \mathrm{T}_\ell(E) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

as a $\mathbb{Q}_\ell[\Gamma_k]$-module determines the elliptic curve up to k-isogeny, assuming k is finitely generated over its prime field [Fal83, Zar76, Mor78]. In addition, the Tate module also contains information about reduction types (c.f. Néron–Ogg–Shafarevich criterion [Sil09, Theorem VII.7.1] and Serre–Tate theorem [ST68, Theorem 1]), endomorphisms (c.f. Tate's theorem [Tat66, Main Theorem]), point counts over finite fields (c.f. Grothendieck–Lefschetz trace formula [Mil13, Theorem 29.4]), etc.

2.2.3 Notation. We use the following notation:

$k : \mathsf{Field}$	A field
$U : \mathsf{Sch}_{\mathrm{Spec}(k)}$	A Noetherian scheme over k
$\ell : \mathbb{N}$	A prime with $\ell \neq \mathrm{char}(k)$
$\Lambda : \mathsf{Ring}^{\mathrm{top}}$	One of $\mathbb{Z}_\ell$, $\mathbb{Q}_\ell$, or $\mathbb{Z}/\ell^n\mathbb{Z}$ for $n : \mathbb{N}$
$U_{\text{ét}} : \mathsf{Site}$	The small étale site of U

2.2.4 Definition. (Mod-ℓ^n Sheaf) For each $n : \mathbb{N}$, a *mod-ℓ^n sheaf* on U is a sheaf of $\mathbb{Z}/\ell^n\mathbb{Z}$-modules on $U_{\text{ét}}$. Denote their (symmetric monoidal $\mathbb{Z}/\ell^n\mathbb{Z}$-linear) category as

$$\mathsf{Sh}(U, \mathbb{Z}/\ell^n\mathbb{Z}) := \mathsf{Sh}(U_{\text{ét}}, \mathsf{Mod}_{\mathbb{Z}/\ell^n\mathbb{Z}}).$$

A mod-ℓ^n sheaf is *lisse* if it is locally constant with finite geometric stalks. We denote by $\mathsf{LocSys}_{\mathbb{Z}/\ell^n\mathbb{Z}}(U)$ the (symmetric monoidal $\mathbb{Z}/\ell^n\mathbb{Z}$-linear) full subcategory of lisse sheaves.

2.2.5 Definition. (ℓ-adic Sheaf) An *(integral) ℓ-adic sheaf* on U is a family $\mathcal{F}$ of sheaves

$$\mathcal{F}_n : \mathsf{Sh}(U, \mathbb{Z}/\ell^n\mathbb{Z})$$

equipped with maps $r_n : \mathcal{F}_{n+1} \to \mathcal{F}_n$ in $\mathsf{Sh}(U, \mathbb{Z}/\ell^{n+1}\mathbb{Z})$, such that

$$\overline{r_n} : \mathcal{F}_{n+1}/\ell^n\mathcal{F}_{n+1} \xrightarrow{\sim} \mathcal{F}_n$$

factoring through r_n is an isomorphism in $\mathsf{Sh}(U, \mathbb{Z}/\ell^{n+1}\mathbb{Z})$, for each $n : \mathbb{N}$. We say that $\mathcal{F}$ is *lisse* if each $\mathcal{F}_n$ is lisse. Denote by $\mathsf{Sh}(U, \mathbb{Z}_\ell)$ the (symmetric monoidal $\mathbb{Z}_\ell$-linear) category of ℓ-adic sheaves on U, and $\mathsf{LocSys}_{\mathbb{Z}_\ell}(U)$ the (symmetric monoidal $\mathbb{Z}_\ell$-linear) full subcategory of lisse sheaves.

2.2.6 Definition. (Rational ℓ-adic Sheaf) Define the category of *rational ℓ-adic sheaves* on U, denoted $\mathsf{Sh}(U, \mathbb{Q}_\ell)$, as the (symmetric monoidal $\mathbb{Q}_\ell$-linear) category where the objects are the same as $\mathsf{Sh}(U, \mathbb{Z}_\ell)$ but the morphisms between two objects $\mathcal{F}, \mathcal{G} : \mathsf{Sh}(U, \mathbb{Q}_\ell)$ are

$$\mathrm{Hom}_{\mathsf{Sh}(U,\mathbb{Q}_\ell)}(\mathcal{F}, \mathcal{G}) := \mathrm{Hom}_{\mathsf{Sh}(U,\mathbb{Z}_\ell)}(\mathcal{F}, \mathcal{G}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

considered as tensoring of $\mathbb{Z}_\ell$-modules. An object of this category is a rational ℓ-adic sheaf on U. There is a canonical functor called *rationalization*

$$(-)_{\mathbb{Q}_\ell} = (-) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell : \mathsf{Sh}(U, \mathbb{Z}_\ell) \to \mathsf{Sh}(U, \mathbb{Q}_\ell), \qquad \mathcal{F} \mapsto \mathcal{F}, \qquad f \mapsto f \otimes 1.$$

A rational ℓ-adic sheaf $\mathcal{F}$ on U is *lisse* if there exists a lisse integral ℓ-adic sheaf $\mathcal{L}$ and an isomorphism $\mathcal{F} \cong \mathcal{L} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. Denote by $\mathsf{LocSys}_{\mathbb{Q}_\ell}(U)$ the (symmetric monoidal $\mathbb{Q}_\ell$-linear) full subcategory of lisse sheaves of $\mathsf{Sh}(U, \mathbb{Q}_\ell)$.

2.2.7. Unfortunately a lot of usual sheaf theoretic constructions do not work for integral and rational ℓ-adic sheaves and require corrections, see Appendix A.1. However, we note that Bhatt and Scholze [BS15] have developed the framework of *pro-étale sites*, allowing ℓ-adic sheaves to be considered as bona fide sheaves on the pro-étale site $U_{\text{pro-ét}}$, which removes most of the clunkiness we had to deal with in the classical setting. For this report, we stick with the classical setting.

2.2.8. Recall the étale fundamental group of a connected U at geometric base point $u_0 : \mathrm{Spec}(\Omega) \to U$ is the profinite group defined by the inverse limit

$$\pi_1^{\text{ét}}(U, u_0) := \varprojlim_{(X,x_0)\to(U,u_0)} \mathrm{Aut}_U(X) : \mathsf{Grp}^{\mathrm{pro}}$$

where the limit is taken over connected finite étale Galois covers $X \to U$ with x_0 a lift of u_0, and the transition map $\varphi : \mathrm{Aut}_U(Y) \to \mathrm{Aut}_U(X)$ for a pointed map $h : (Y, y_0) \to (X, x_0)$ between conneced finite étale Galois covers is given by $h \circ \sigma = \varphi(\sigma) \circ h$, for $\sigma : \mathrm{Aut}_U(Y)$.

2.2.9. Alternatively, one can construct the étale fundamental group via the *fibre functor*

$$\omega_{u_0} : \acute{\mathsf{Et}}^{\mathrm{fin}}(U) \to \mathsf{Set}^{\mathrm{fin}}, \qquad E \mapsto \mathrm{Hom}_U(\mathrm{Spec}(\Omega), E)$$

where $\acute{\mathsf{Et}}^{\mathrm{fin}}(U)$ is the category of finite étale covers $X \to U$ with morphisms over U. We can then construct the étale fundamental group as

$$\pi_1^{\text{ét}}(U, u_0) = \mathrm{Aut}(\omega_{u_0}) : \mathsf{Grp}^{\mathrm{pro}}$$

with topology given by the subspace topology induced by

$$\pi_1^{\text{ét}}(U, u_0) \hookrightarrow \prod_{E : \text{Ét}^{\text{fin}}(U)} \text{Sym}(\omega_{u_0}(E))$$

where each symmetric group is given the discrete topology.

2.2.10. Just as in 2.1.7, for connected U and a geometric base point $u_0 : \text{Spec}(\Omega) \to U$, we have a symmetric monoidal Λ-linear equivalence of categories

$$\text{LocSys}_\Lambda(U) \simeq^{\otimes} \text{Rep}_\Lambda^{\text{fin}}\big(\pi_1^{\text{ét}}(U, u_0)\big)$$

If Λ is $\mathbb{Q}_\ell$ or $\mathbb{F}_\ell$, the *algebraic monodromy group* of a lisse sheaf $\mathcal{F}$ at u_0 is the Zariski closure of the image of its associated representation $G_{\mathcal{F}} := \rho_{\mathcal{F}, u_0}(\pi_1^{\text{ét}}(U, u_0))^{\text{Zar}}$.

2.2.11. We remark that in the context of the p-adic analogue of lisse ℓ-adic sheaves, where monodromy representations are unavailable, Tannakian formalism (see Appendix A.5) is required to define algebraic monodromy groups; c.f. [Cre92, §2].

2.2.12. Let $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$ for $n : \mathbb{N}$, and $\mathcal{F} : \text{LocSys}_\Lambda(U)$. Let $E \to U$ be the finite étale cover representing the underlying sheaf of sets of $\mathcal{F}$, we have a (continuous) representation

$$\rho_{\mathcal{F}} : \pi_1^{\text{ét}}(U, u_0) \to \text{Aut}_{\text{Mod}_\Lambda}(\mathcal{F}_{u_0})$$

with the action by $\sigma : \pi_1^{\text{ét}}(U, u_0)$ induced by

$$\sigma_E : \omega_{u_0}(E) \to \omega_{u_0}(E)$$

under the canonical identification

$$\mathcal{F}_{u_0} = \varinjlim_{\substack{q : V \to U \text{ étale} \\ v : \text{Spec}(\Omega) \to V \text{ geo. pt.} \\ q \circ v = u_0}} \mathcal{F}(V) \xrightarrow{\sim} \omega_{u_0}(E), \qquad [(V, v), s] \mapsto s \circ v$$

where $(s : V \to E) : \text{Hom}_U(V, E) = \mathcal{F}(V)$ is representing a section.

2.2.13. Conversely, given a (continuous) representation in $\text{Rep}_\Lambda^{\text{fin}}(\pi_1^{\text{ét}}(U, u_0))$

$$\rho : \pi_1^{\text{ét}}(U, u_0) \to \text{Aut}_{\text{Mod}_\Lambda}(M)$$

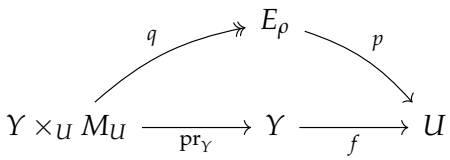
Let $N = \text{Ker}(\rho)$, which is open by continuity, so $G = \pi_1^{\text{ét}}(U, u_0)/N$ is finite and ρ factorizes

$$\pi_1^{\text{ét}}(U, u_0) \twoheadrightarrow G \xrightarrow{\bar{\rho}} \text{Aut}_{\text{Mod}_\Lambda}(M).$$

By Grothendieck's Galois theory (see Appendix A.2), the $\pi_1^{\text{ét}}(U, u_0)$-set G corresponds to a finite étale cover $f : Y \to U$ with $\omega_{u_0}(Y) \cong G$ as $\pi_1^{\text{ét}}(U, u_0)$-sets. By the identification we have a free transitive right G-action on $\omega_{u_0}(Y)$ equivariant with the $\pi_1^{\text{ét}}(U, u_0)$-action. By Grothendieck's Galois theory, this corresponds to a deck transformation of Y transitive and free on each geometric fibre, which makes Y an étale G-torsor. By finiteness of M, set $M_U := (\coprod_{m : M} U \to U) : \text{Sch}_U$ and form the space called the *contracted product*

$$E_\rho := Y \times^G M := (Y \times_U M_U)/G \cong \left(\coprod_{m : M} Y\right) \Big/ G$$

where G acts by $(y, m) \cdot g = (yg, \overline{\rho}(g)^{-1}m)$. There is a unique map $p : E_\rho \to U$ factoring through the G-invariant composite map



where q is the quotient map. Thus we found a finite étale cover

$$(E_\rho \xrightarrow{p} U) : \acute{\mathsf{E}}\mathsf{t}^{\mathrm{fin}}(U)$$

Moreover, the module operations on M are G-equivariant and descend to addition, zero, and scalar-multiplication morphisms over U, making E_ρ a Λ-module scheme over U. By taking its sheaf of sections

$$\mathcal{F}_\rho(V) := \mathrm{Hom}_U(V, E_\rho) = \{s : V \to E_\rho \mid p \circ s = q\} : \mathsf{Mod}_\Lambda,$$

for $q : V \to U$ étale, we obtain our desired lisse sheaf $\mathcal{F}_\rho : \mathsf{LocSys}_\Lambda(U)$. To see that $\mathcal{F}_\rho$ is étale-locally constant, note that pulling back by $f : Y \to U$ gives

$$\mathcal{F}_\rho|_Y \cong f^*\mathcal{F}_\rho \cong \mathrm{Hom}_Y(-, Y \times_U E_\rho) \cong \mathrm{Hom}_Y\left(-, \coprod_{m\,:\,M} Y\right) \cong \underline{M}_Y.$$

2.2.14. The case for $\Lambda = \mathbb{Z}_\ell$ is achieved by applying the construction for $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$ and assembling each $n : \mathbb{N}$, and the case for $\Lambda = \mathbb{Q}_\ell$ is by rationalizing, see A.1.3.1 and A.1.3.2.

2.2.15 Lisse Sheaves from Torsors. Given a finite group $G : \mathsf{Grp}^{\mathrm{fin}}$, a finitely generated Λ-module $M : \mathsf{Mod}_\Lambda^{\mathrm{top}}$, a continuous representation $\rho : G \to \mathrm{Aut}_\Lambda(M)$ with G given the discrete topology, and a finite étale right G-torsor $f : Y \to U$, we construct a local system $\mathcal{L}_\rho : \mathsf{LocSys}_\Lambda(U)$ by following the same procedure in 2.2.13, A.1.3.1, and A.1.3.2. For $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$, as in 2.2.13, form the cover

$$E_\rho := Y \times^G M = (Y \times_U M_U)/G \xrightarrow{p} U,$$

where $M_U := (\coprod_{m\,:\,M} U \to U) : \mathsf{Sch}_U$ and G acts by $(y, m) \cdot g = (yg, \rho(g)^{-1}m)$, then take the sheaf of sections to obtain the lisse sheaf

$$\mathcal{L}_\rho := \mathrm{Hom}_U(-, E_\rho) : \mathsf{LocSys}_\Lambda(U).$$

Assembling and rationalizing give the cases $\Lambda = \mathbb{Z}_\ell$ (see A.1.4.1) and $\Lambda = \mathbb{Q}_\ell$ (see A.1.4.2).

2.2.16 Example. We give some concrete examples of 2.2.15.

1. Let $N : \mathbb{N}$ be invertible in k, and assume that k contains all N-th roots of unity. Let $Y = U = \mathbb{G}_m/k$, $G = \mu_N(k)$, and the G-torsor $f : Y \to U$ by $z \mapsto z^N$ with the right multiplication action $z \cdot \zeta = z\zeta$ for $\zeta : \mu_N(k)$. Let $K/\mathbb{Q}_\ell$ be a finite field extension, and let $M = K$ if $\Lambda = \mathbb{Q}_\ell$, $M = \mathcal{O}_K$ if $\Lambda = \mathbb{Z}_\ell$, and $M = \mathcal{O}_K/\ell^n\mathcal{O}_K$ if $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$. Choose a character $\chi : \mu_N(k) \to M^\times$; then we get the representation

$$\rho_\chi : G \to \mathrm{Aut}_\Lambda(M), \qquad \rho_\chi(\zeta)(m) := \chi(\zeta)m.$$

   The lisse sheaf we obtain from these data, denoted $\mathcal{L}_\chi$, is the *Kummer sheaf* of χ.

2. Assume positive characteristic $\mathrm{char}(k) = p \neq \ell$. Let $Y = U = \mathbb{G}_a/k$, $G = \mathbb{F}_p$, and let $f : Y \to U$ be the G-torsor $z \mapsto z^p - z$ under the translation action $z \cdot a = z + a$. Let $K/\mathbb{Q}_\ell$ be a finite field extension containing a primitive p-th root of unity, and let $M = K$ if $\Lambda = \mathbb{Q}_\ell$, $M = \mathcal{O}_K$ if $\Lambda = \mathbb{Z}_\ell$, and $M = \mathcal{O}_K/\ell^n\mathcal{O}_K$ if $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$. Choose a (additive) character $\psi : \mathbb{F}_p \to M^\times$; then we get the representation

$$\rho_\psi : G \to \mathrm{Aut}_\Lambda(M), \qquad \rho_\psi(a)(m) := \psi(a)m.$$

The lisse sheaf we obtain from these data, denoted $\mathcal{L}_\psi$, is the *Artin–Schreier sheaf* of ψ.

2.2.17 Example. A rich source of lisse sheaves is smooth proper families, as in 2.1.8. Let

$$f : E \to U$$

be a smooth proper morphism of k-schemes. The geometric fibres $E_{u_0} := E \times_U \operatorname{Spec}\Omega$ at geometric points $u_0 : \operatorname{Spec}\Omega \to U$ are smooth proper Ω-schemes. The higher direct image

$$\mathbf{R}^q f_* \underline{\Lambda}_E$$

are lisse on U for each $q : \mathbb{N}$ by smooth proper base change [Mil13, Theorem 20.2 and §19], i.e. for $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$, every geometric point u_0 has an étale neighbourhood (V, v_0) with

$$(\mathbf{R}^q f_* \underline{\Lambda}_E)|_V \cong \underline{\mathrm{H}^q_{\text{ét}}(E_{u_0}, \underline{\Lambda})}_V.$$

Proper base change identifies the geometric stalks as

$$\left(\mathbf{R}^q f_* \underline{\Lambda}_E\right)_{u_0} \cong \mathrm{H}^q_{\text{ét}}(E_{u_0}, \underline{\Lambda}).$$

3 Perverse Sheaves

3.1 Constructible Sheaves

3.1.1. We return to our example in 1.3.4 of the double cover, but with the origin included

$$f : \mathbb{C} \to \mathbb{C}, \qquad z \mapsto z^2$$

Its sheaf of sections $\mathcal{F} : \mathsf{Sh}(\mathbb{C}, \mathsf{Set})$ is no longer a local system of sets due to the ramification at 0. Indeed the fibres at 0 have cardinality 1 where the fibres at a nonzero point have cardinality 2. However, we can stratify the space $\mathbb{C}$ into

$$\mathbb{C} = \mathbb{C}^\times \sqcup \{0\}$$

Then $\mathcal{F}$ is a local system of sets when restricted to each stratum. This is essentially the idea behind constructible sheaves, we want to allow sheaves that are not necessarily locally constant, but are locally constant when restricted to each stratum of a stratification. This way, we can allow for mild failures of local constancy such as ramification.

3.1.2 Notation. We use the following notation

$k : \mathsf{Field}$	A field
$\ell : \mathbb{N}$	A prime with $\ell \neq \mathrm{char}(k)$

$\Lambda : \mathsf{Ring}^{\mathrm{top}}$	One of $\mathbb{Z}_\ell$, $\mathbb{Q}_\ell$, or $\mathbb{Z}/\ell^n\mathbb{Z}$ for $n : \mathbb{N}$
$X, Y : \mathsf{Sch}_{\mathrm{Spec}(k)}$	Separated schemes of finite type over k
$\pi : X \to \mathrm{Spec}(k)$	The structure morphism of X
$S : \mathsf{Sch}_{\mathrm{Spec}(k)}$	A finite reduced closed subscheme of X
$i : S \hookrightarrow X$	The closed inclusion map
$m : \mathbb{N}$	The number of k^{sep}-valued points of S, $m := \#S(k^{\mathrm{sep}})$
$U : \mathsf{Sch}_{\mathrm{Spec}(k)}$	The complementary open subscheme $X \setminus S$
$j : U \hookrightarrow X$	The open inclusion map

3.1.3 Definition. (Stratification) A *stratification* of X is a finite family of nonempty locally closed subschemes $i_\alpha : X_\alpha \hookrightarrow X$ indexed by $\alpha : I$ such that they are

1. disjoint: $X_\alpha \times_X X_\beta = \emptyset$, for $\alpha, \beta : I$ with $\alpha \neq \beta$,
2. jointly surjective: the induced map $\coprod_{\alpha:I} X_\alpha \to X$ is surjective.

3.1.4 Definition. (Algebraically Constructible Sheaves) A *(algebraically) constructible* sheaf on X is a sheaf $\mathcal{F} : \mathsf{Sh}(X, \Lambda)$ such that there exists a stratification $i_\alpha : X_\alpha \to X$ indexed by $\alpha : I$ with the restriction $\mathcal{F}|_{X_\alpha}$ lisse for all $\alpha : I$. Denote by $\mathsf{Sh}_{\mathrm{c}}(X, \Lambda)$ the full subcategory of constructible sheaves.

3.1.5. Let $U = \mathbb{C}^\times$ and $X = \mathbb{C}$. In the situation in 3.1.1, suppose we have $\mathcal{F} : \mathsf{LocSys}_\Lambda(U)$ that we wish to extend to a constructible sheaf on the whole space X in the simplest way. One way to do so is set the stalks at the singularities to be zero

$$(j_!\mathcal{F})_u \cong \begin{cases} \mathcal{F}_u, & u \neq 0, \\ 0, & u = 0, \end{cases} \qquad u : \mathbb{C},$$

This is *extension by zero*, the compactly-supported pushforward of inclusion $j : U \hookrightarrow X$.

3.1.6 Definition. (Compactly-supported Pushforward) Let $f : X \to Y$ be a morphism. For a constructible sheaf $\mathcal{F} : \mathsf{Sh}_{\mathrm{c}}(X, \Lambda)$, we define the *compactly-supported pushforward* $f_!\mathcal{F} : \mathsf{Sh}_{\mathrm{c}}(Y, \Lambda)$ along f, for each coefficient Λ. For $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$, for each étale $V \to Y$, set

$$(f_!\mathcal{F})(V) := \left\{ s : \mathcal{F}(X_V) \;\middle|\; \mathrm{supp}(s) \hookrightarrow X_V \xrightarrow{f_V} V \text{ is proper} \right\},$$

For integral and rational ℓ-adic sheaves, apply the usual corrections, see A.1.5.1. For an étale morphism $a : V \to X$ of finite type, the *compactly-supported sections* of $\mathcal{F}$ over V are

$$\begin{aligned} \Gamma_c(V, \mathcal{F}) &:= \Gamma(\mathrm{Spec}(k), (\pi \circ a)_! a^* \mathcal{F}) \\ &= \{ s : \Gamma(V, a^*\mathcal{F}) \mid \mathrm{supp}(s) \text{ is proper over } k \} : \mathsf{Mod}_\Lambda. \end{aligned}$$

The *compactly-supported cohomology* are defined by

$$\mathbf{R}\Gamma_c(V, \mathcal{F}) := \mathrm{H}^q\big(\mathbf{R}\Gamma_c(V, \mathcal{F})\big).$$

for $q : \mathbb{Z}$; c.f. [Stacks, Definition 63.12.1, Tag 0GJZ].

3.1.7. Forgetting supports induces a natural morphism ([Stacks, Remark 63.9.5])

$$\mathbf{R}f_!K \longrightarrow \mathbf{R}f_*K$$

for every $K : \mathsf{D}_c^b(X, \Lambda)$, which is an isomorphism when f is proper.

3.1.8. There is a canonical short exact sequence for each constructible sheaf $\mathcal{F} : \mathsf{Sh}_c(X, \Lambda)$

$$0 \longrightarrow j_!j^*\mathcal{F} \xrightarrow{\varepsilon} \mathcal{F} \xrightarrow{\eta} i_*i^*\mathcal{F} \longrightarrow 0,$$

where ε is the counit of $j_! \dashv j^*$ and η is the unit of $i^* \dashv i_*$ [Stacks, Tag 095L]. This exact sequence yields the Euler–Poincaré and Grothendieck–Ogg–Shafarevich formulas stated in A.3.2 and A.3.3.

3.2 ℓ-adic Six-Functor Formalism

3.2.1 Notation. We use the following notation

$k : \mathsf{Field}$	A field
$X, Y : \mathsf{Sch}_{\mathrm{Spec}(k)}$	Separated schemes of finite type over k
$f : X \to Y$	A morphism over k
$\pi : X \to \mathrm{Spec}(k)$	The structure morphism of X
$\ell : \mathbb{N}$	A prime with $\ell \neq \mathrm{char}(k)$
$\Lambda : \mathsf{Ring}^{\mathrm{top}}$	One of $\mathbb{F}_\ell$ or $\mathbb{Q}_\ell$

3.2.2. Denote by $\mathsf{D}_c^b(X, \Lambda)$ the category of bounded constructible derived Λ-sheaves on X. We describe in this section Grothendieck's profound yoga of six functors, which are these functors on $\mathsf{D}_c^b(X, \Lambda)$. These are the pullback f^*, derived pushforward $\mathbf{R}f_*$, derived tensor product $- \otimes_\Lambda^{\mathbf{L}} -$, derived sheaf Hom $\mathbf{R}\mathcal{H}om_\Lambda(-,-)$, derived pushforward with proper support $\mathbf{R}f_!$, and the exceptional pullback $f^!$, which will be defined below.

3.2.3 Verdier Duality. The *relative Verdier duality theorem*, c.f. [SGA4III, Exp. XVIII, Theorem 3.1.4] and [Kat96, §2.2.1], asserts that $\mathbf{R}f_!$ admits a right adjoint,

$$f^! : \mathsf{D}_c^b(Y, \Lambda) \longrightarrow \mathsf{D}_c^b(X, \Lambda),$$

called the *exceptional pullback*, characterized by natural isomorphisms

$$\mathrm{Hom}_{\mathsf{D}_c^b(Y,\Lambda)}(\mathbf{R}f_!K, L) \cong \mathrm{Hom}_{\mathsf{D}_c^b(X,\Lambda)}(K, f^!L)$$

for each $K : \mathsf{D}_c^b(X, \Lambda)$ and $L : \mathsf{D}_c^b(Y, \Lambda)$. We can upgrade the adjunction to

$$\mathbf{R}\mathcal{H}om_\Lambda(\mathbf{R}f_!K, L) \cong \mathbf{R}f_*\mathbf{R}\mathcal{H}om_\Lambda(K, f^!L),$$

replacing Hom-sets with internal Homs; c.f. [Stacks, Lemma 63.11.5].

3.2.4 Definition. (Verdier Dual) The *dualizing complex* of X is

$$\omega_X := \pi^!\Lambda.$$

For $K : \mathsf{D}_c^b(X, \Lambda)$, its *Verdier dual* is

$$\mathbb{D}_X K := \mathbf{R}\mathcal{H}om_\Lambda(K, \omega_X).$$

This defines a contravariant functor on $\mathsf{D}_c^b(X, \Lambda)$; c.f. [Kat96, §2.2.1].

3.2.5. The Verdier duality functor is *involutive*: there is a natural isomorphism

$$K \xrightarrow{\sim} \mathbb{D}_X \mathbb{D}_X K$$

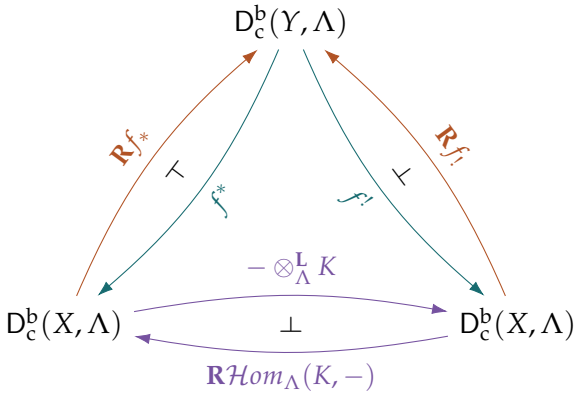
for every $K : \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X, \Lambda)$; c.f. [Kat96, §2.2.1].

3.2.6. Verdier duality exchanges the six functors in the following ways:

$$\begin{aligned}
\mathbb{D}_Y(\mathbf{R}f_* K) &\cong \mathbf{R}f_!(\mathbb{D}_X K), \\
\mathbb{D}_Y(\mathbf{R}f_! K) &\cong \mathbf{R}f_*(\mathbb{D}_X K), \\
\mathbb{D}_X(f^* M) &\cong f^!(\mathbb{D}_Y M), \\
\mathbb{D}_X(f^! M) &\cong f^*(\mathbb{D}_Y M), \\
\mathbb{D}_X(K \otimes^{\mathbf{L}}_{\Lambda} L) &\cong \mathbf{R}\mathcal{H}om_{\Lambda}(K, \mathbb{D}_X L), \\
\mathbb{D}_X\big(\mathbf{R}\mathcal{H}om_{\Lambda}(K, L)\big) &\cong K \otimes^{\mathbf{L}}_{\Lambda} \mathbb{D}_X L,
\end{aligned}$$

for $K, L : \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X, \Lambda)$ and $M : \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(Y, \Lambda)$. c.f. [Kat96, §2.1.2(4)–(5) and §2.2.1].

3.2.7. Therefore, there are three pairs of adjoints in the six functors.



for each $K : \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X, \Lambda)$, where matching color arrows are Verdier duals.

3.2.8. Other properties in six functor formalism are recalled in Appendix A.4.

3.3 Middle Extension and Middle Convolution

3.3.1. The Riemann–Hilbert correspondence is a symmetric monoidal equivalence

$$\begin{aligned}
\mathsf{Bun}^{\nabla}(X) &\simeq^{\otimes} \mathsf{LocSys}_{\mathbb{C}}(X), \\
(\mathcal{E}, \nabla) &\longmapsto \ker \nabla
\end{aligned}$$

where $X : \mathsf{Man}_{\mathbb{C}}$ and $\mathsf{Bun}^{\nabla}(X)$ is the category of finite-rank holomorphic vector bundles $\mathcal{E}$ equipped with an integrable connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X$. Perverse sheaves [BBD82] arose as a generalization of the right hand side, from generalizing the left hand side to regular holonomic $\mathcal{D}$-modules.

$$\mathsf{Mod}^{\mathrm{rh}}_{\mathcal{D}_X} \simeq \mathsf{Perv}(X, \mathbb{C}),$$

where $\mathsf{Mod}^{\mathrm{rh}}_{\mathcal{D}_X}$ denotes the category of regular holonomic left $\mathcal{D}_X$-modules and $\mathsf{Perv}(X, \mathbb{C})$ the category of complex perverse sheaves. This generalization allows for singularities, as in 3.1.1.

3.3.2 Notation. We use the following notation

$k : \mathsf{Field}$	A field
$\ell : \mathbb{N}$	A prime with $\ell \neq \mathrm{char}(k)$
$\Lambda : \mathsf{Ring}^{\mathrm{top}}$	One of $\mathbb{F}_\ell$ or $\mathbb{Q}_\ell$
$X, Y : \mathsf{Sch}_{\mathrm{Spec}(k)}$	Separated schemes of finite type over k
$f : X \to Y$	A morphism over k
$S : \mathsf{Sch}_{\mathrm{Spec}(k)}$	A finite reduced closed subscheme of X
$i : S \hookrightarrow X$	The closed inclusion
$U : \mathsf{Sch}_{\mathrm{Spec}(k)}$	The complementary open subscheme $X \setminus S$
$j : U \hookrightarrow X$	The open inclusion

3.3.3 Definition. (Perverse Sheaves) An object $K : \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X, \Lambda)$ is *semiperverse* if

$$\dim \mathrm{Supp}\, \mathcal{H}^n(K) \leq -n \qquad \text{for every } n : \mathbb{Z},$$

It is *perverse* if both K and its Verdier dual $\mathbb{D}_X K$ are semiperverse. The subcategories

$$ {}^p\mathsf{D}^{\leq 0} := \left\{ K : \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X, \Lambda) \;\middle|\; K \text{ is semiperverse} \right\},$$
$$ {}^p\mathsf{D}^{\geq 0} := \left\{ K : \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X, \Lambda) \;\middle|\; \mathbb{D}_X K \text{ is semiperverse} \right\}$$

form the *middle perverse t-structure*. Its heart $\mathsf{Perv}(X, \Lambda) := {}^p\mathsf{D}^{\leq 0} \cap {}^p\mathsf{D}^{\geq 0}$ is the abelian category of *perverse sheaves*; c.f. [Kat96, §2.3.1 and §2.3.1.1].

3.3.4 Example. (Punctual Sheaf) A constructible sheaf $\mathcal{F} : \mathsf{Sh}_{\mathrm{c}}(X, \Lambda)$ is *punctual* if we can choose S and U such that the unit of the adjunction $i^* \dashv i_*$,

$$\eta^i_{\mathcal{F}} : \mathcal{F} \longrightarrow i_* i^* \mathcal{F},$$

is an isomorphism. Equivalently, its support is a finite set of closed points. Viewed as a complex concentrated in degree 0, it is perverse, i.e. $\mathcal{F} : \mathsf{Perv}(X, \Lambda)$; c.f. [Kat96, §2.3.6].

3.3.5 Example. (Middle Extension Sheaf) Suppose X is a smooth geometrically connected curve. A constructible sheaf $\mathcal{F} : \mathsf{Sh}_{\mathrm{c}}(X, \Lambda)$ is a *middle extension sheaf* if, we can choose S and U such that the restriction $j^*\mathcal{F}$ is lisse and the unit of the adjunction $j^* \dashv j_*$,

$$\eta^j_{\mathcal{F}} : \mathcal{F} \longrightarrow j_* j^* \mathcal{F},$$

is an isomorphism. Here j_* is ordinary direct image. Viewed as a complex concentrated in degree -1, it is perverse i.e. $\mathcal{F}[1] : \mathsf{Perv}(X, \Lambda)$. c.f. [Kat96, §2.0.3 and §2.3.6].

3.3.6 Middle Direct Image. For $K : \mathsf{Perv}(X, \Lambda)$ such that $\mathbf{R}f_! K, \mathbf{R}f_* K : \mathsf{Perv}(Y, \Lambda)$, define

$$f_{!*} K := \mathrm{Im}_{\mathsf{Perv}(Y,\Lambda)} \left(\mathbf{R}f_! K \longrightarrow \mathbf{R}f_* K \right),$$

called the *middle direct image*, where the arrow is the forgetting supports from 3.1.7. This upgrades to a functor $f_{!*} : \mathsf{Perv}(X, \Lambda) \to \mathsf{Perv}(Y, \Lambda)$ and commutes with Verdier duality: $\mathbb{D}_Y(f_{!*} K) \cong f_{!*}(\mathbb{D}_X K)$. If j is an affine open immersion, $j_{!*}$ is fully faithful; c.f. [Kat96, §2.3.3.1].

3.3.7 Perverse Sheaves on a Smooth Curve. Suppose X is a smooth geometrically connected curve. An object $K : \mathsf{D}_c^b(X, \Lambda)$ is perverse if and only if

1. $\mathcal{H}^n(K) = 0$ for $n \neq -1, 0$;
2. $\mathcal{H}^{-1}(K)$ has no nonzero punctual subsheaf;
3. $\mathcal{H}^0(K)$ is punctual.

As a consequence, choose U such that $j^*\mathcal{H}^{-1}(K)$ is lisse, then K admits a filtration

$$0 \subseteq \mathrm{Coker}\Big(\mathcal{H}^{-1}(K) \longrightarrow j_* j^* \mathcal{H}^{-1}(K)\Big) \subseteq \mathcal{H}^{-1}(K)[1] \subseteq K,$$

in $\mathsf{Perv}(X, \Lambda)$. Its quotients are punctual, a (shifted) middle extension sheaf, and punctual.

3.3.8 External Tensor Products. Let $\mathrm{pr}_1 : X \times_k Y \to X$ and $\mathrm{pr}_2 : X \times_k Y \to Y$ be the projections. The *external tensor product* is the bifunctor

$$\begin{aligned} \boxtimes_\Lambda : \mathsf{D}_c^b(X, \Lambda) \times \mathsf{D}_c^b(Y, \Lambda) &\longrightarrow \mathsf{D}_c^b(X \times_k Y, \Lambda), \\ (K, L) &\longmapsto K \boxtimes_\Lambda L := \mathrm{pr}_1^* K \otimes_\Lambda^{\mathbf{L}} \mathrm{pr}_2^* L. \end{aligned}$$

It is exact in each variable, and there is a natural isomorphism

$$\mathbb{D}_{X \times_k Y}(K \boxtimes_\Lambda L) \cong (\mathbb{D}_X K) \boxtimes_\Lambda (\mathbb{D}_Y L).$$

c.f. [Kat96, §2.2.2].

3.3.9. Let $a : \mathbb{A}_k^1 \times_k \mathbb{A}_k^1 \longrightarrow \mathbb{A}_k^1$ be the addition morphism. For $K, L : \mathsf{D}_c^b(\mathbb{A}_k^1, \Lambda)$, recall

$$\begin{aligned} K *_! L &:= \mathbf{R}a_!(K \boxtimes_\Lambda L), \\ K *_* L &:= \mathbf{R}a_*(K \boxtimes_\Lambda L). \end{aligned}$$

is their !-*convolution* and $*$-*convolution*.

3.3.10 Middle Convolution. For $K, L : \mathsf{Perv}(\mathbb{A}_k^1, \Lambda)$ their *middle convolution* is

$$K *_{!*} L := \mathrm{Im}_{\mathsf{Perv}(\mathbb{A}_k^1, \Lambda)} (K *_! L \longrightarrow K *_* L) = a_{!*}(K \boxtimes_\Lambda L),$$

where we assume $K *_! L$ and $K *_* L$ are perverse; c.f. [Kat96, §2.6.2].

4 Monodromy in Arithmetic

4.1 Open Image Theorems in Arithmetic

4.1.1 Notation. We use the following notation:

$k : \mathsf{Field}$	A field
$\ell : \mathbb{N}$	A prime with $\ell \neq \mathrm{char}(k)$
$A : \mathsf{Var}_k$	An abelian variety
$g : \mathbb{N}$	The dimension of A

Γ_k : $\mathsf{Grp}^{\mathrm{pro}}$ The absolute Galois group of k

4.1.2 Galois Representations arising from Abelian Varieties. Applying 2.2.17 to the structure morphism $f : A \to \mathrm{Spec}(k)$ with $q = 1$ and $\Lambda = \mathbb{Q}_\ell$ gives the lisse sheaf $\mathbf{R}^1 f_* \underline{\mathbb{Q}_\ell}_A$, whose geometric stalk carries the $2g$-dimensional Galois representation

$$\rho^1_{A,\ell} : \Gamma_k \longrightarrow \mathrm{Aut}_{\mathbb{Q}_\ell}\big(\mathrm{H}^1_{\text{ét}}(A_{k^{\mathrm{sep}}}, \underline{\mathbb{Q}_\ell})\big).$$

By a canonical Γ_k-equivariant identification $\mathrm{H}^1_{\text{ét}}(A_{k^{\mathrm{sep}}}, \underline{\mathbb{Q}_\ell}) \cong \mathrm{V}_\ell(A)^\vee$, this is precisely the dual of the Tate module.

4.2 Middle Convolution with Kummer Sheaves

4.3 Big Monodromy of Hyperelliptic Curves

Appendix

A.1 ℓ-adic Corrections

A.1.1 Artin–Rees Strictification

A.1.1.1. Many common sheaf constructions for integral ℓ-adic sheaves can be taken level-wise such as pullbacks, direct sums, and tensor products, but unfortunately there are also many common sheaf constructions that require corrections, since an operation could produce a family $\mathcal{A}$ that does not satisfy

$$\mathcal{A}_{n+1}/\ell^n\mathcal{A}_{n+1} \cong \mathcal{A}_n$$

Whenever such a representative exists, we replace the family with an object that does satisfy this condition and is isomorphic to it in the Artin–Rees category [SGA5, Exp. V]. Explicitly, we replace $\mathcal{A}$ with $\mathcal{A}'$ where

$$\mathcal{A}'_n = \mathcal{B}_N/\ell^n\mathcal{B}_N, \qquad \text{where} \qquad \mathcal{B}_n := \bigcap_{m\geq n} \operatorname{Im}(\mathcal{A}_m \longrightarrow \mathcal{A}_n),$$

for each $n : \mathbb{N}$, with $N : \mathbb{N}$ sufficiently large, dependent on n. Constructions that require such corrections include kernels, images, sheaf Hom, and duals.

A.1.1.2. Some constructions of integral ℓ-adic sheaves require taking inverse limits. For an integral ℓ-adic sheaf $\mathcal{F}$, these include:

Name	Formula	Assumptions
Geometric stalks	$\mathcal{F}_{u_0} := \varprojlim_n (\mathcal{F}_n)_{u_0}$	$u_0 : \operatorname{Spec}(\Omega) \to U$ $\Omega : \mathsf{Field},\ \text{separably closed}$
Sections	$\Gamma(V,\mathcal{F}) := \varprojlim_n \Gamma(V,\mathcal{F}_n)$	$V : U_{\text{ét}}$
Sheaf cohomology	$\mathrm{H}^q(U,\mathcal{F}) := \varprojlim_n \mathrm{H}^q(U_{\text{ét}},\mathcal{F}_n)$	$q : \mathbb{N}$
Morphism modules	$\operatorname{Hom}(\mathcal{F},\mathcal{G}) \cong \varprojlim_n \operatorname{Hom}_{\mathsf{Sh}(U,\mathbb{Z}/\ell^n\mathbb{Z})}(\mathcal{F}_n,\mathcal{G}_n)$	$\mathcal{G} : \mathsf{Sh}(U,\mathbb{Z}_\ell)$

A.1.2 Rationalization

Rational ℓ-adic sheaves require corrections by rationalizing the integral constructions. For a rational ℓ-adic sheaf $\mathcal{F}_{\mathbb{Q}_\ell} = \mathcal{F} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ with $\mathcal{F}$ an integral ℓ-adic sheaf, constructions of $\mathcal{F}_{\mathbb{Q}_\ell}$ are usually done by first applying the corresponding (corrected) constructions for $\mathcal{F}$, and then rationalizing. For example:

Name	Formula	Assumptions
Geometric stalks	$(\mathcal{F}_{\mathbb{Q}_\ell})_{u_0} := \mathcal{F}_{u_0} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$	$u_0 : \operatorname{Spec}(\Omega) \to U$ $\Omega : \mathsf{Field},\ \text{separably closed}$
Sections	$\Gamma(V,\mathcal{F}_{\mathbb{Q}_\ell}) := \Gamma(V,\mathcal{F}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$	$V : U_{\text{ét}}$

Name	Formula	Assumptions
Sheaf cohomology	$\mathrm{H}^q(U, \mathcal{F}_{\mathbb{Q}_\ell}) := \mathrm{H}^q(U, \mathcal{F}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$	$q : \mathbb{N}$
Kernel	$\mathrm{Ker}(f \otimes 1) := (\mathrm{Ker}\, f) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$	$f : \mathcal{F} \to \mathcal{G}$ in $\mathsf{Sh}(U, \mathbb{Z}_\ell)$
Image	$\mathrm{Im}(f \otimes 1) := (\mathrm{Im}\, f) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$	$f : \mathcal{F} \to \mathcal{G}$ in $\mathsf{Sh}(U, \mathbb{Z}_\ell)$
Cokernel	$\mathrm{Coker}(f \otimes 1) := (\mathrm{Coker}\, f) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$	$f : \mathcal{F} \to \mathcal{G}$ in $\mathsf{Sh}(U, \mathbb{Z}_\ell)$
Coimage	$\mathrm{Coim}(f \otimes 1) := (\mathrm{Coim}\, f) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$	$f : \mathcal{F} \to \mathcal{G}$ in $\mathsf{Sh}(U, \mathbb{Z}_\ell)$
Pullback	$g^*(\mathcal{F}_{\mathbb{Q}_\ell}) := (g^*\mathcal{F})_{\mathbb{Q}_\ell}$	$g : V \to U$ $V : \mathsf{Sch}_{\mathrm{Spec}(k)}$, Noetherian
Direct sum	$\mathcal{F}_{\mathbb{Q}_\ell} \oplus \mathcal{G}_{\mathbb{Q}_\ell} := (\mathcal{F} \oplus \mathcal{G})_{\mathbb{Q}_\ell}$	$\mathcal{G} : \mathsf{Sh}(U, \mathbb{Z}_\ell)$
Tensor product	$\mathcal{F}_{\mathbb{Q}_\ell} \otimes_{\mathbb{Q}_\ell} \mathcal{G}_{\mathbb{Q}_\ell} := (\mathcal{F} \otimes_{\mathbb{Z}_\ell} \mathcal{G})_{\mathbb{Q}_\ell}$	$\mathcal{G} : \mathsf{Sh}(U, \mathbb{Z}_\ell)$
Sheaf Hom	$\mathcal{H}om_{\mathbb{Q}_\ell}(\mathcal{F}_{\mathbb{Q}_\ell}, \mathcal{G}_{\mathbb{Q}_\ell}) := (\mathcal{H}om_{\mathbb{Z}_\ell}(\mathcal{F}, \mathcal{G}))_{\mathbb{Q}_\ell}$	$\mathcal{G} : \mathsf{Sh}(U, \mathbb{Z}_\ell)$
Dual	$(\mathcal{F}_{\mathbb{Q}_\ell})^\vee := (\mathcal{F}^\vee)_{\mathbb{Q}_\ell}$	$\mathcal{F}^\vee := \mathcal{H}om_{\mathbb{Z}_\ell}(\mathcal{F}, \mathbb{Z}_\ell)$

A.1.3 Lisse Sheaves

A.1.3.1. Let $\Lambda = \mathbb{Z}_\ell$. Given a continuous representation

$$\rho : \pi_1^{\text{ét}}(U, u_0) \to \mathrm{Aut}_{\mathbb{Z}_\ell}(M)$$

on a finitely generated $\mathbb{Z}_\ell$-module $M : \mathsf{Mod}_\Lambda^{\mathrm{top}}$ with its ℓ-adic topology, apply 2.2.13 to the induced representations ρ_n on $M_n := M/\ell^n M$ for $n \geq 1$, obtaining sheaves $\mathcal{F}_n$. The reductions $M_{n+1} \twoheadrightarrow M_n$ are equivariant, and, by Grothendieck's Galois theory, they induce transition maps $r_n : \mathcal{F}_{n+1} \to \mathcal{F}_n$ satisfying

$$\mathcal{F}_{n+1}/\ell^n \mathcal{F}_{n+1} \cong \mathcal{F}_n,$$

Thus we obtain the integral lisse ℓ-adic sheaf $\mathcal{F}$. Conversely, for a lisse sheaf $\mathcal{F}$, the stalk $\mathcal{F}_{u_0}$ has a natural $\pi_1^{\text{ét}}(U, u_0)$-action by taking the $\pi_1^{\text{ét}}(U, u_0)$-action with the induce map on each level, and assembling them together.

A.1.3.2. Let $\Lambda = \mathbb{Q}_\ell$. Given a continuous representation $\rho : \pi_1^{\text{ét}}(U, u_0) \to \mathrm{Aut}_{\mathbb{Q}_\ell}(M)$ on a finite-dimensional $\mathbb{Q}_\ell$-vector space M, choose a ρ-stable $\mathbb{Z}_\ell$-lattice $L \subset M$, which exists because the image of ρ is compact [Ser92, Part II, Ch. IV, Appendix 1, Theorem 1]. Apply A.1.3.1 to the action on L, obtaining an integral lisse sheaf $\mathcal{F}_L$, and set $\mathcal{F}_\rho := \mathcal{F}_L \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. Conversely, choose an integral lisse model $\mathcal{G}$ of a rational lisse sheaf $\mathcal{F} \cong \mathcal{G} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. Extend the stalk action from A.1.3.1 by scalars:

$$\mathcal{F}_{u_0} = \mathcal{G}_{u_0} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell, \qquad \rho_{\mathcal{F}}(\gamma) := \rho_{\mathcal{G}}(\gamma) \otimes \mathrm{id}_{\mathbb{Q}_\ell}.$$

A.1.4 Lisse Sheaves from Torsors

A.1.4.1. For $\Lambda = \mathbb{Z}_\ell$, as in A.1.3.1, apply the finite-coefficient construction to $M_n := M/\ell^n M$ with its induced G-action to obtain $\mathcal{L}_n$. The G-equivariant reductions $M_{n+1} \twoheadrightarrow M_n$ induce maps of associated covers and hence transition maps $r_n : \mathcal{L}_{n+1} \to \mathcal{L}_n$ by composition of sections. These satisfy $\mathcal{L}_{n+1}/\ell^n \mathcal{L}_{n+1} \cong \mathcal{L}_n$, giving $\mathcal{L}_\rho : \mathsf{LocSys}_{\mathbb{Z}_\ell}(U)$.

A.1.4.2. For $\Lambda = \mathbb{Q}_\ell$, as in A.1.3.2, choose a G-stable $\mathbb{Z}_\ell$-lattice $L \subset M$ and apply the integral construction L to obtain $\mathcal{L}_L$, then rationalize: $\mathcal{L}_\rho := \mathcal{L}_L \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell : \mathsf{LocSys}_{\mathbb{Q}_\ell}(U)$.

A.1.5 Compactly-supported Pushforward

A.1.5.1. For $\Lambda = \mathbb{Z}_\ell$, write $\mathcal{F} = (\mathcal{F}_n)_{n:\mathbb{N}}$ and define $f_!\mathcal{F}$ by applying A.1.1.1 to $(f_!\mathcal{F}_n)_{n:\mathbb{N}}$. For $\Lambda = \mathbb{Q}_\ell$, let $\mathcal{L} : \mathsf{Sh}(X, \mathbb{Z}_\ell)$ with $\mathcal{F} \cong \mathcal{L} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, and set $f_!\mathcal{F} := (f_!\mathcal{L}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$.

A.2 Grothendieck's Galois Theory

A.3 Euler Characteristics of Curves

A.3.1. We use the notation of 3.1.2.

A.3.2 Euler–Poincaré Formula. Let $k = \mathbb{C}$ and X a smooth projective connected curve of genus $g : \mathbb{N}$. For each $\mathcal{F} : \mathsf{Sh}_c(X^{\mathrm{an}}, \mathbb{C})$ with pullback $\mathcal{L} := (j^{\mathrm{an}})^*\mathcal{F}$ a local system of rank $r : \mathbb{N}$, the then

$$\chi(X^{\mathrm{an}}, \mathcal{F}) = r(2 - 2g - m) + \sum_{s \in S(\mathbb{C})} \dim_{\mathbb{C}} \mathcal{F}_s.$$

where $\mathcal{F}_s$ denotes the stalk at s. If $\mathcal{F} = (j^{\mathrm{an}})_*\mathcal{L}$, then

$$\mathcal{F}_s \cong \mathrm{Ker}(T_s - \mathrm{id}_{\mathcal{L}_{u_s}}) = \{v : \mathcal{L}_{u_s} \mid T_s(v) = v\}$$

is the space of local monodromy invariants, where u_s lies in a sufficiently small punctured disc around s and T_s is the monodromy around s based at u_s; c.f. [Kat96, §1.0.2 and proof of Theorem 1.1.2]. The arithmetic analogue is the Grothendieck–Ogg–Shafarevich formula.

A.3.3 Grothendieck–Ogg–Shafarevich Formula. Assume k is perfect and X is a smooth projective geometrically connected curve of genus $g : \mathbb{N}$. For finitely generated $M : \mathsf{Mod}_\Lambda$,

$$d_\Lambda(M) := \begin{cases} \mathrm{rank}_\Lambda M, & \Lambda = \mathbb{Z}_\ell \text{ or } \mathbb{Q}_\ell, \\ \mathrm{length}_\Lambda M, & \Lambda = \mathbb{Z}/\ell^n\mathbb{Z}. \end{cases}$$

Let $\mathcal{F} : \mathsf{Sh}_c(X, \Lambda)$ have lisse pullback $\mathcal{L} := j^*\mathcal{F}$, and put $r := d_\Lambda(\mathcal{L}_{u_0})$ for a geometric point u_0 of U. Define the *Euler characteristic* by

$$\chi(X_{k^{\mathrm{sep}}}, \mathcal{F}_{k^{\mathrm{sep}}}) := \sum_{q=0}^{2} (-1)^q \, d_\Lambda\big(\mathrm{H}^q(X_{k^{\mathrm{sep}}}, \mathcal{F}_{k^{\mathrm{sep}}})\big).$$

Then

$$\chi(X_{k^{\mathrm{sep}}}, \mathcal{F}_{k^{\mathrm{sep}}}) = r(2 - 2g - m) + \sum_{s \in S(k^{\mathrm{sep}})} \big(d_\Lambda(\mathcal{F}_s) - \mathrm{Sw}_s(\mathcal{L})\big),$$

where $\mathcal{F}_s$ is the geometric stalk at s and $\mathrm{Sw}_s(\mathcal{L})$ is the *Swan conductor* of $\mathcal{L}$ at s, defined as follows. Let V_s be the geometric generic stalk of $\mathcal{L}$ on the punctured strict henselian neighbourhood of s in $X_{k^{\mathrm{sep}}}$, with local inertia group I_s. Writing I_s^t for its upper-numbering ramification groups, set

$$\mathrm{Sw}_s(\mathcal{L}) := \int_0^\infty \big(d_\Lambda(V_s) - d_\Lambda(V_s^{I_s^t})\big) \, \mathrm{d}t.$$

It vanishes when the local inertia action is tame. c.f. [SGA5, Exp. X, §7].

A.4 Properties of the Six Functors

A.4.1. We use the notation of 3.2.1.

A.4.2 Smooth Purity. If $f : X \to Y$ is smooth of pure relative dimension $d : \mathbb{N}$, there is a natural isomorphism [SGA4III, Exp. XVIII, Theorem 3.2.5]

$$f^! M \cong f^* M(d)[2d]$$

in $M : \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(Y, \Lambda)$, where (d) is the Tate twist. If X is smooth of pure dimension d over k,

$$\omega_X = \pi^! \Lambda \cong \Lambda(d)[2d].$$

c.f. [Kat96, §2.1.2(6) and §2.2.1].

A.4.3 Poincaré Duality. Suppose $f : X \to Y$ is smooth of pure relative dimension $d : \mathbb{N}$. For a lisse sheaf $\mathcal{F} : \mathsf{LocSys}_\Lambda(X)$, there is an isomorphism natural in $\mathcal{F}$,

$$\mathbf{R}f_*(\mathcal{F}^\vee(d))[2d] \cong \mathbf{R}\mathcal{H}om_\Lambda(\mathbf{R}f_!\mathcal{F}, \underline{\Lambda}_Y).$$

This follows from A.4.2 and relative Verdier duality; c.f. [Stacks, Lemma 63.11.5].

A.4.4 Projection Formula. There is a natural isomorphism

$$\mathbf{R}f_!(K \otimes^{\mathbf{L}}_\Lambda f^* L) \cong \mathbf{R}f_! K \otimes^{\mathbf{L}}_\Lambda L.$$

for $K : \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X, \Lambda)$ and $L : \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(Y, \Lambda)$; c.f. [Stacks, Lemma 63.10.7, Tag 0GL5]. If L is lisse of finite rank, there is also a natural isomorphism

$$\mathbf{R}f_*(K \otimes^{\mathbf{L}}_\Lambda f^* L) \cong \mathbf{R}f_* K \otimes^{\mathbf{L}}_\Lambda L;$$

c.f. [Stacks, Lemma 21.50.1, Tag 0944].

A.4.5 Base Change. Consider a Cartesian square:

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

For $K : \mathrm{D}^b_c(X, \Lambda)$, there are natural comparison morphisms

$$g^* \mathbf{R}f_\varepsilon K \longrightarrow \mathbf{R}f'_\varepsilon(g'^* K), \qquad \varepsilon : \{!, *\}.$$

which are isomorphisms if $\varepsilon = !$ OR if $\varepsilon = *$ and f is proper or g is smooth; c.f. [Stacks, Theorems 59.91.11 and 59.89.2, and Lemma 63.9.4]. Additionally, if $\varepsilon = !$ OR if $\varepsilon = *$ and f is smooth and proper, then $\mathbf{R}^q f_\varepsilon \mathcal{F}$ is lisse for $q : \mathbb{N}$ and $\mathcal{F} : \mathsf{LocSys}_\Lambda(X)$.

A.5 Tannakian Formalism

A.5.1. In 1939, Tannaka showed that compact groups can be recovered from their finite-dimensional unitary representations [Tan39]. Grothendieck later introduced the Tannakian formalism [Mil25, §1], which provides a general framework for what was discussed in § 2.1 and § 2.2.

A.5.2 Definition. (Neutral Tannakian Category) A *neutral tannakian category* C over $\Lambda : \mathsf{Field}$ is a rigid symmetric monoidal abelian Λ-linear category, equipped with an exact faithful Λ-linear symmetric monoidal functor called the *fibre functor*

$$\omega : \mathsf{C} \to \mathsf{Vect}_{\Lambda}^{\mathrm{fin}}$$

and is such that the scalar map $\Lambda \xrightarrow{\sim} \mathrm{End}(\mathbf{1})$ is an isomorphism.

A.5.3. Let $\Lambda : \mathsf{Field}$ be one of $\mathbb{C}$, $\mathbb{Q}_\ell$, or $\mathbb{F}_\ell$ with $\ell \neq \mathrm{char}(k)$ for $k : \mathsf{Field}$, then the category $\mathsf{LocSys}_\Lambda(U)$ of local systems or lisse sheaves on $U : \mathsf{Man}_{\mathbb{C}}$ or $U : \mathsf{Var}_k$ is a neutral tannakian category. Its fibre functor is

$$\omega_{u_0} : \mathsf{LocSys}_\Lambda(U) \to \mathsf{Vect}_{\Lambda}^{\mathrm{fin}}, \qquad \mathcal{F} \mapsto \mathcal{F}_{u_0}$$

at a base point $u_0 : U$ or geometric point $u_0 : \mathrm{Spec}(\Omega) \to U$.

A.5.4 Definition. (Tannakian Group) Let C be a neutral Tannakian category over $\Lambda : \mathsf{Field}$ with fibre functor $\omega : \mathsf{C} \to \mathsf{Vect}_{\Lambda}^{\mathrm{fin}}$, then its *Tannakian group* is the functor $G : \mathsf{Alg}_\Lambda \to \mathsf{Grp}$

$$G(R) = \mathrm{Aut}^{\otimes}(\omega_R), \qquad \omega_R(\mathcal{F}) := \omega(\mathcal{F}) \otimes_\Lambda R, \qquad \mathcal{F} : \mathsf{C}$$

where $\mathrm{Aut}^{\otimes}$ denotes the natural automorphisms compatible with tensor products and the tensor unit. Moreover, the *monodromy group* of $\mathcal{F} : \mathsf{C}$ is the functor $G_{\mathcal{F}} : \mathsf{Alg}_\Lambda \to \mathsf{Grp}$

$$G_{\mathcal{F}}(R) = \mathrm{Aut}^{\otimes}(\omega_R \mid_{\langle \mathcal{F} \rangle_{\otimes}})$$

where $\langle \mathcal{F} \rangle_{\otimes}$ is the strictly full subcategory of C consisting of subquotients of finite direct sums of tensor products of $\mathcal{F}$ and $\mathcal{F}^\vee$, including the empty tensor product $\mathbf{1}$. We remark that this subcategory is also neutral Tannakian over the same field.

A.5.5 Theorem. (Tannakian Reconstruction Theorem) *Let* C *be a neutral Tannakian category over* $\Lambda : \mathsf{Field}$ *with fibre functor* $\omega : \mathsf{C} \to \mathsf{Vect}_{\Lambda}^{\mathrm{fin}}$*, then its Tannakian group* $G : \mathsf{Alg}_\Lambda \to \mathsf{Grp}$ *is representable as an affine group scheme. Moreover* ω *induces a* Λ*-linear symmetric monoidal equivalence of categories*

$$\mathsf{C} \simeq^{\otimes} \mathsf{Rep}_{\Lambda}^{\mathrm{alg,fin}}(G)$$

where the right hand side is the category of algebraic representations of an affine group scheme.

Proof. c.f. [DM82, Theorem 2.11]. □

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