Trace of Frobenius

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Outline

Motivation

Question. How many solutions does the equation

$$E: \; y^2 + y = x^3 + x + 1$$

have in $\mathbb{F}_{2^n}$, for $n \geq 1$ (counting a point at infinity)?

Denoting $N_n = \#E(\mathbb{F}_{2^n})$ for $n \geq 1$

$$N_1 = 1$$
$$N_2 = 5$$
$$N_3 = 13$$
$$N_4 = 25$$
$$N_5 = 41$$
$$N_6 = 65$$
$$N_7 = 113$$
$$N_8 = 225$$

Is there a pattern?

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With sufficiently little social life, one can spot a pattern

$$N_5 = 5(25) - 10(13) + 10(5) - 4(1) = 5N_4 - 10N_3 + 10N_2 - 4N_1$$
$$N_6 = 5(41) - 10(25) + 10(13) - 4(5) = 5N_5 - 10N_4 + 10N_3 - 4N_2$$
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Conjecture. The numbers N_n satisfy

$$N_n = 5N_{n-1} - 10N_{n-2} + 10N_{n-3} - 4N_{n-4}$$

for $n \geq 5$.

Assuming this conjecture, we can easily find a formula for N_n.

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Consider the $\mathbb{C}$-vector space V of sequences $(a_n)_{n \in \mathbb{Z}}$ in $\mathbb{C}$ satisfying

$$a_n = 5a_{n-1} - 10a_{n-2} + 10a_{n-3} - 4a_{n-4}$$

for all $n \in \mathbb{Z}$.

There is an obvious isomorphism

$$\varphi : V \to \mathbb{C}^4 \qquad (a_n)_{n \in \mathbb{Z}} \mapsto (a_1, a_2, a_3, a_4)$$

so $\dim V = 4$.

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Now consider the linear operator

$$F : V \to V \qquad F((a_n)_{n\in\mathbb{Z}}) = (a_{n+1})_{n\in\mathbb{Z}}$$

By the recurrence relation, one easily sees that

$$F^4 = 5F^3 - 10F^2 + 10F - 4I$$

The minimal polynomial must be degree 4, so char. poly. of F is

$$\lambda^4 - 5\lambda^3 + 10\lambda^2 - 10T + 4 = (\lambda - 1)(\lambda - 2)(\lambda^2 - 2\lambda + 2)$$

so the eigenvalues are $1, 2$ and the two roots α, β of $\lambda^2 - 2\lambda + 2$

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The eigenvectors are $(1)_{n\in\mathbb{Z}}, (2^n)_{n\in\mathbb{Z}}, (\alpha^n)_{n\in\mathbb{Z}}, (\beta^n)_{n\in\mathbb{Z}}$, which are linearly independent, hence forming an eigenbasis. Thus

$$N_n = A + B2^n + C\alpha^n + D\beta^n$$

Solving a linear system of equations for $n = 1, 2, 3, 4$, we get

$$N_n = 1 + 2^n - \alpha^n - \beta^n$$

The coefficients are ± 1: **this is an important hint**.

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We decompose $V = H^0 \oplus H^1 \oplus H^2$ where

$$H^0 = \mathrm{Span}\{(1)_{n\in\mathbb{Z}}\}$$
$$H^1 = \mathrm{Span}\{(\alpha^n)_{n\in\mathbb{Z}}, (\beta^n)_{n\in\mathbb{Z}}\}$$
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Then by rearraging the formulas

$$\begin{aligned} N_n &= 1^n - (\alpha^n + \beta^n) + 2^n \\ &= \mathrm{tr}(F^n \mid H^0) - \mathrm{tr}(F^n \mid H^1) + \mathrm{tr}(F^n \mid H^2) \end{aligned}$$

where $\mathrm{tr}(F^n \mid H^i)$ represents trace of F^n restricted to H^i.

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Theorem (Lefschetz). Let X be a compact complex manifold and $f : X \to X$ a holomorphic map with isolated simple fixed points; let

$$\Lambda_f := \sum_{k \geq 0} (-1)^k \operatorname{tr}(f_* \mid H_k(X, \mathbb{Q}))$$

which we call its *Lefschetz number*, then

$$\Lambda_f = \#\{x \in X : f(x) = x\}$$

counts the number of fixed points of f.

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Define the *arithmetic Frobenius* for $\overline{\mathbb{F}_q}$

$$F : \overline{\mathbb{F}_q} \to \overline{\mathbb{F}_q} \qquad x \mapsto x^q$$

We note that for any variety X over $\mathbb{F}_q$,

$$X(\mathbb{F}_{q^n}) = X(\overline{\mathbb{F}_q})^{\mathrm{Frob}_q^n}$$

where $\mathrm{Frob}_q : X_{\overline{\mathbb{F}_q}} \to X_{\overline{\mathbb{F}_q}}$ is the *geometric Frobenius*, defined by

$$\mathrm{Frob}_q = \mathrm{id}_X \times_{\mathbb{F}_q} F^{-1} : X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q} \to X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}$$

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where $\mathrm{Frob}_q : X_{\overline{\mathbb{F}_q}} \to X_{\overline{\mathbb{F}_q}}$ is the *geometric Frobenius*, defined by

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base changing along the inverse of the arithmetic Frobenius.

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Conjecture (Weil, 1949). Suppose that X is a smooth projective geometrically connected variety of dimension d over $\mathbb{F}_q$, and define $N_n := \#X(\mathbb{F}_{q^n})$ for $n \geq 1$, then there exists finitely many algebraic integers $\lambda_{i,j}$ such that for all $n \geq 1$

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$$\zeta_X(T) = \exp\left(\sum_{n\geq 1} \frac{\#X(\mathbb{F}_q^n)}{n} T^n\right) \in \mathbb{Q}[[T]]$$

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Grothendieck–Lefschetz Trace Formula

Let ℓ be a prime such that $\ell \neq \mathrm{char}(\mathbb{F}_q)$

Definition. An *ℓ-adic sheaf* (of modules) on a noetherian scheme $X/\mathbb{F}_q$ is a projective system $\{\mathcal{F}_n\}_{n\geq 1}$ of étale sheaves in the sense that $\mathcal{F}_n$ is a sheaf of $\mathbb{Z}/\ell^n\mathbb{Z}$-modules on $X_{\text{ét}}$ for all n, and there are morphisms $\mathcal{F}_{n+1} \to \mathcal{F}_n$ of sheaves of $\mathbb{Z}/\ell^{n+1}\mathbb{Z}$-modules (regarding $\mathcal{F}_n$ via restriction of scalars) for all n, such that the induced maps

$$\mathcal{F}_{n+1} \otimes_{\mathbb{Z}/\ell^{n+1}\mathbb{Z}} \mathbb{Z}/\ell^n\mathbb{Z} \longrightarrow \mathcal{F}_n$$

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Definition. Let $\{\mathcal{F}\}_{n\geq 1}$ be an ℓ-adic sheaf on $X/\mathbb{F}_q$ a noetherian scheme, then we say it is

1. *lisse* if each $\mathcal{F}_n$ is locally constant with finite stalks,
2. *constructible* if there exists a *stratification* $X = \coprod_{i=0}^{N} X_i$ into locally closed subschemes such that $\mathcal{F}_n$ is locally constant on each X_i with finite stalks for all n.

Definition. Let $\mathcal{F} = \{\mathcal{F}\}_{n\geq 1}$ be a constructible ℓ-adic sheaf, let

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Theorem (Grothendieck–Lefschetz). Suppose X is a smooth projective geometrically connected variety of dimension d over $\mathbb{F}_q$, then for any $n \geq 1$, we have

$$\#X(\mathbb{F}_{q^n}) = \sum_{i=1}^{2d}(-1)^i \mathrm{tr}(\mathrm{Frob}_q^n \mid \mathrm{H}^i_{\text{ét}}(X, \mathbb{Q}_\ell))$$

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Sketch of Proof

Definition. Let $X/\mathbb{F}_q$ be a scheme and $f : X \to X$ a morphism. Let the *graph* of f be

$$\Gamma_f : X \to X \times X \qquad \Gamma_f = (\mathrm{id}, f)$$

and define the *diagonal* as

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Definition. Let $X/\mathbb{F}_q$ be a scheme and $f : X \to X$ a morphism. Let the *graph* of f be

$$\Gamma_f : X \to X \times X \qquad \Gamma_f = (\mathrm{id}, f)$$

and define the *diagonal* as

$$\Delta : X \to X \times X \qquad \Delta = (\mathrm{id}, \mathrm{id})$$

We then have $\#X(\mathbb{F}_{q^n}) = \#(\Gamma_{\mathrm{Frob}_q^n} \cap \Delta)$.

Let Z, W be integral closed subschemes of a scheme X over an algebraically closed field of complementary dimension. Suppose they intersect transversely i.e. for all $p \in Z \cap W$,

$$\mathrm{T}_p(Z) + \mathrm{T}_p(W) = \mathrm{T}_p(X)$$

then their *intersection product* $[Z] \cdot [W]$ is a linear combination of their intersection points with coefficients all 1. Its degree

$$\deg([Z] \cdot [W]) = \sum_{p \in Z \cap W} 1 = \#(Z \cap W)$$

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Definition. Let $d \in \mathbb{Z}$, the *Tate twist* of the ℓ-adic sheaf $\mathbb{Z}_\ell$

$$\mathbb{Z}_\ell(d) = \begin{cases} \mathbb{Z}_\ell(1)^{\otimes d} & \text{if } d > 1 \\ \varprojlim_n \mu_{\ell^n} & \text{if } d = 1 \\ \mathbb{Z}_\ell & \text{if } d = 0 \\ \mathrm{Hom}_{\mathbb{Z}_\ell}(\mathbb{Z}_\ell(1), \mathbb{Z}_\ell) & \text{if } d = -1 \\ \mathbb{Z}_\ell(-1)^{\otimes d} & \text{if } d < -1 \end{cases}$$

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Let X be a smooth scheme over a field, ℓ a prime invertible in X, and $0 \leq r \leq \dim X$ an integer, then there is *ℓ-adic cycle class map*

$$\mathrm{cl}_X^r : \mathrm{CH}^r(X) \to \mathrm{H}^{2r}_{\text{ét}}(X, \mathbb{Z}_\ell(r))$$

which associates to each integral closed codimension r subschemes (up to rational equivalence) a class in $\mathrm{H}^{2r}_{\text{ét}}(X, \mathbb{Z}_\ell(r))$.

Moreover, for $[Z]$ codimension s and $[W]$ codimension t,

$$\mathrm{cl}_X^{s+t}([Z] \cdot [W]) = \mathrm{cl}_X^s([Z]) \smile \mathrm{cl}_X^t([W])$$

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Theorem (Poincare Duality). With same settings. The map

$$\mathrm{H}^i_{\text{ét}}(X, \mathbb{Z}_\ell(r)) \times \mathrm{H}^{2d-i}_{\text{ét}}(X, \mathbb{Z}_\ell(d-r)) \xrightarrow{\smile} \mathrm{H}^{2d}_{\text{ét}}(X, \mathbb{Z}_\ell(d)) \xrightarrow{\int_X} \mathbb{Z}_\ell$$

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Theorem (Kunneth formula). Let X, Y be schemes over a common base field, $\mathcal{F}, \mathcal{G}$ be ℓ-adic sheaves. The *Kunneth map*

$$\bigoplus_{i+j=n} \mathrm{H}^i(X, \mathcal{F}) \otimes \mathrm{H}^j(Y, \mathcal{G}) \xrightarrow{\alpha\otimes\beta\mapsto\pi_1^*\alpha\smile\pi_2^*\beta} \mathrm{H}^{i+j}(X \times Y, \mathcal{F} \boxtimes \mathcal{G})$$

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Under the identification to $\bigoplus_{i=0}^{2d} \mathrm{End}(\mathrm{H}^i(X, \mathbb{Q}_\ell))$,

$$\mathrm{cl}([\Delta]) \sim \mathrm{id} \qquad \text{and} \qquad \mathrm{cl}([\Gamma_f]) \sim f^*$$

Picking basis $(e_{i,\alpha})$ of $\mathrm{H}^i(X, \mathbb{Q}_\ell)$ for each i. One computes

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Let $f = \mathrm{Frob}_q^n$. The pièce de résistance is

$$\begin{aligned}
\#X(\mathbb{F}_{q^n}) &= \#(\Delta \cap \Gamma_f) \\
&= \deg([\Delta] \cdot [\Gamma_f]) \\
&= \int_{X \times X} \mathrm{cl}([\Delta] \cdot [\Gamma_f]) \\
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\end{aligned}$$

Let $f = \mathrm{Frob}_q^n$. The pièce de résistance is

$$\begin{aligned}
\#X(\mathbb{F}_{q^n}) &= \#(\Delta \cap \Gamma_f) \\
&= \deg([\Delta] \cdot [\Gamma_f]) \\
&= \int_{X \times X} \mathrm{cl}([\Delta] \cdot [\Gamma_f]) \\
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Using the graded commutativity of box tensors

$$(\alpha \boxtimes \beta) \smile (\gamma \boxtimes \delta) = (-1)^{\deg(\beta)\deg(\gamma)} (\alpha \boxtimes \gamma) \smile (\beta \boxtimes \delta)$$

and the *cohomological Fubini formula*

$$\int_{X \times Y} (\alpha \boxtimes \beta) = \left(\int_X \alpha \right) \left(\int_Y \beta \right)$$

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The trace $\int_X \alpha$ is only nonzero when α is a top cohomological class, which is only when $i + j = 2d$.

$$\begin{aligned}
\#X(\mathbb{F}_{q^n}) &= \sum_{i,\alpha,j,\beta} (-1)^{ij} \int_X e_{i,\alpha} \smile f^*(e_{j,\beta}) \int_X e_{i,\alpha}^\vee \smile e_{j,\beta}^\vee \\
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We now write $f^*(e_{i,\alpha}) = \sum_\beta m_{\beta,\alpha} e_{i,\beta}$, so $(m_{\beta,\alpha})$ is the matrix of f^* on the vector space $\mathrm{H}^i(X, \mathbb{Q}_\ell)$. Then

$$\begin{aligned}\sum_\alpha \int_X e_{i,\alpha} \smile f^*(e_{i,\alpha}) &= \sum_{\alpha,\beta} m_{\beta,\alpha} \int_X (e_{i,\alpha} \smile e_{i,\beta}) \\ &= \sum_\alpha m_{\alpha,\alpha} \\ &= \mathrm{tr}(f^* \mid \mathrm{H}^i(X, \mathbb{Q}_\ell))\end{aligned}$$

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Applications and Future Directions

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